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Inverse limits and full families

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Abstract

In this paper we investigate inverse limits on $[0,1]$ using a single bonding map chosen from a Full family (one-parameter family of C^1 unimodal maps). Our investigation makes use of the renormalization operator utilized by Feigenbaum to explain the universal way in which Full families transition from simple to complicated dynamics. Among other results, we show that up through the Feigenbaum value the inverse limit is hereditarily decomposable with a fascinating pattern in the appearance of topological $\sin(\frac{1}{x})$-curves. Approaching the Feigenbaum value from above we see a similar pattern in the appearance of the Brouwer–Janiszewski–Knaster indecomposable continuum.

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1. Introduction

Inverse limits, besides being of intrinsic interest to topologists, can often be used to represent attractors of dynamical systems. For example, the inverse limit space with a single full unimodal bonding map is homeomorphic to the attracting set of Smale's horseshoe. Williams [26] and Block [7] were the first to address the relationship between inverse limits and attractors and many others have since followed. These efforts have generated an increasing interest in the topological properties of inverse limit spaces with unimodal bonding maps.

Barge and Martin in [1] showed that there is a strong relationship between the dynamics of the bonding map and the topology of the corresponding inverse limit. In the same work, Barge and Martin also showed that the inverse limits corresponding to unimodal maps with finite kneading sequences of different lengths are not homeomorphic due to the fact

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that they have a different number of endpoints. Holte [16] utilized kneading theory to show that two unimodal bonding maps with the same finite kneading sequence produce homeomorphic inverse limits. Only recently has it been shown that two inverse limits with bonding maps having different kneading sequences of the same finite length are not homeomorphic [19] (see also [4] and [23]).

Many of these results have been concentrated on the *tent family*,

$$T_\lambda = \begin{cases} \lambda x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ \lambda(1-x) & \text{if } \frac{1}{2} \leqslant x \leqslant 1, \end{cases} \quad 0 \leqslant \lambda \leqslant 2,$$

or the *logistic family*

$$f_\lambda(x) = \lambda x(1-x), \quad 0 \leqslant \lambda \leqslant 4.$$

This is of no surprise as these two families are the most investigated and well understood examples of one-parameter families of interval maps. An important difference between the tent family and logistic family is that only the latter is an example of a *Full family* (see Section 7). This paper investigates the topology of the inverse limit generated by a single bonding map chosen from a Full family.

Our interest in Full families is two fold. First, Full families is the setting for Feigenbaum's celebrated Universality Theory. Secondly, apart from the logistic family, the author has found little in the literature concerning the resulting inverse limit space as the parameter within a Full family varies.

We will see that in a Full family of $\mathcal{C}^1$unimodal maps with BC maximal (see Section 3) and not of length a power of 2 there exist sequences

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \lambda_\infty \leftarrow \cdots < \mu_2 < \mu_1 < \mu_0$$

such that the critical point corresponding to λ_n is periodic of period 2^n and the critical point corresponding to μ_n is periodic of period $2^n\,|BC|$ (where $|BC|$ denotes the length of BC). The sequence $\{\lambda_n\}$ represents what is commonly called the *period doubling route to chaos*. By investigating the logistic family, Feigenbaum equipped with only a pocket calculator made a remarkable discovery:

$$\lim_{n\to\infty} \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+2}-\lambda_{n+1}} = \delta = 4.6692106\ldots$$

and is identical for all such systems undergoing this period doubling. Feigenbaum [13] went on to propose an explanation for the universality of δ which was inspired by the renormalization group theory in statistical mechanics.

Barge and Ingram [3] investigated inverse limit spaces using a single bonding map chosen from the logistic family at various parameter values. They revealed a number of striking patterns that occur within the corresponding inverse limits at parameter values below, above, and at the Feigenbaum value, λ_∞. Using kneading theory and the renormalization operator introduced by Feigenbaum, we generalize certain of their results to Full families where a negative Schwarzian derivative is not assumed.

Denoting the inverse limit with unimodal bonding map f by $\varprojlim(I, f)$, some of the results of this paper can be summarized as follows: For parameter value λ such that f_λ has kneading sequence below that of f_{λ_∞}, $\varprojlim(I, f_\lambda)$ is hereditarily decomposable with topological $\sin(\frac{1}{x})$-curves as the dominant subcontinua (Theorems 13 and 14). As a new result even for the logistic family, we show that for the sequence $\{\mu_n\}$ mentioned above, $\varprojlim(I, f_{\mu_{n+1}})$ is a ray limiting on two homeomorphic copies of $\varprojlim(I, f_{\mu_n})$ intersecting a common endpoint (Theorem 16).

2. Preliminaries

Let $X_0, X_1, \ldots$ be a sequence of metric spaces and $f_0, f_1, \ldots$ be a sequence of maps (continuous functions) such that $f_i : X_{i+1} \to X_i$ for each i. Define the *inverse limit* of the inverse sequence (X_i, f_i) by the following:

$$\varprojlim(X_i, f_i) = \left\{ \underline{x} = (x_0, x_1, \ldots) \in \prod_{i=0}^{\infty} X_i \mid f_i(x_{i+1}) = x_i, \text{ for } i = 0, 1, \ldots \right\}$$

with metric given by $d(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$, where d_i is a metric for X_i bounded by one. Each map f_i is called a *bonding map*.

Throughout this paper, an *arc* denotes any topological space homeomorphic to $I = [0, 1]$ and a *continuum* is a nonempty, compact, connected metric space. A continuum is *decomposable* if it is the union of two of its proper subcontinua, otherwise, it is *indecomposable*. For a set A, $|A|$ will denote the number of elements in A. Following [2], we say that a map $f : [a, b] \to [a, b]$ is *nearly Markov* with respect to $A_1, A_2, \ldots, A_n$ if $A_1, A_2, \ldots, A_n$ are disjoint closed subintervals of $[a, b]$ such that the following conditions hold:

(1) $a \in A_1$ and $b \in A_n$,
(2) $f(\bigcup_{i=1}^{n} A_i) \subset \operatorname{int}(\bigcup_{i=1}^{n} A_i)$,
(3) $\operatorname{diam} f^k(A_i) \to 0$ as $k \to \infty$ for $i = 1, 2, \ldots, n$,
(4) f is one-to-one on each component of $[a, b] - \bigcup_{i=1}^{n} A_i$.

Two functions f and g are *topologically conjugate* if there exists a homeomorphism h such that $h \circ f = g \circ h$. It is well known that $\varprojlim(X_i, f_i)$ is a continuum provided each X_i is a continuum. We will be most interested in the inverse limit when each $X_i = I$ and each f_i is the same bonding map f. In this situation, we write $\varprojlim(I, f)$ for the inverse limit.

Suppose J and K are two closed intervals with maps $f : J \to J$ and $g : K \to K$. We will make use of the following well-known results.

Theorem 1. *If f is a homeomorphism, then $\varprojlim(J, f)$ is an arc.*

Theorem 2. *If f and g are topologically conjugate, then $\varprojlim(J, f)$ is homeomorphic to $\varprojlim(K, g)$.*

3. Unimodal maps and kneading theory

A continuous function $f:[a,b]\to[a,b]$ is called a *unimodal map* if $f(a)=f(b)=a$ and there exist a $c\in(a,b)$ such that f is strictly increasing on $[a,c]$ and strictly decreasing on $[c,b]$. For the most part, we will work with $[a,b]=I$. A unimodal map f is called full-unimodal if $f(c)=b$. If $x\in[a,b]$, we define the *itinerary of* x by

$$I(x)=a_0a_1\cdots \quad \text{where } a_i=\begin{cases} L & \text{if } f^i(x)<c,\\ C & \text{if } f^i(x)=c,\\ R & \text{if } f^i(x)>c,\end{cases}$$

with the convention that $I(x)$ is of finite length if $f^i(c)=c$ for some i. Notice that a unimodal map f induces a shift map σ on sequences by $I(f(x))=\sigma(I(x))$, where $\sigma(a_0a_1\cdots)=a_1a_2\cdots$ (if $I(f(x))=C$, $\sigma(I(x))$ is undefined). If we define an order on the symbols L, R, and C by $L<C<R$, then it can be extended to an order on sequences as follows: If $A=a_0a_1\cdots$ and $B=b_0b_1\cdots$ are two different finite or infinite sequences, there is a smallest integer i with $a_i\neq b_i$. We call a finite sequence *odd* if it contains an odd number of Rs and *even* otherwise. We then define $A\prec B$ if, and only if, either $a_i<b_i$ and $a_0a_1\cdots a_{i-1}$ is even or $a_i>b_i$ and $a_0a_1\cdots a_{i-1}$ is odd.

An itinerary $I(x)$ is said to be *maximal* if $\sigma^n(I(x))\preceq I(x)$ for all $n\geqslant 0$ for which $\sigma^n(I(x))$ is defined. The fact that $I(f(c))$ is maximal and $\sigma^n(I(x))\preceq I(f(c))$ for all $x\in I$ and all $n\geqslant 0$ leads to the following definition. The *kneading sequence* of a unimodal map f with critical point c, denoted $k(f)$, is defined as $k(f)=I(f(c))$. A sequence $a_0a_1\cdots$ is *admissible* if it is infinite and contains only Ls and Rs or is a finite sequence of Ls and Rs ending with a C. The length of a sequence B is denoted by $|B|$.

Let $\widehat{L}=R$, $\widehat{R}=L$, and $\widehat{C}=C$. For a finite sequence A of Ls and Rs and admissible sequence $B=b_0b_1\cdots$ we define the $*$-operator as follows:

(1) If A is even and B is infinite, $A*B=Ab_0Ab_1\cdots$.
(2) If A is even and $B=b_0b_1\cdots b_{n-1}C$, $A*B=Ab_0Ab_1\cdots Ab_{n-1}AC$.
(3) If A is odd and B is infinite, $A*B=A\hat{b}_0A\hat{b}_1\cdots$.
(4) If A is odd and $B=b_0b_1\cdots b_{n-1}C$, $A*B=A\hat{b}_0A\hat{b}_1\cdots A\hat{b}_{n-1}AC$.

The $*$-operator will be important when we consider the renormalization operator $\Re$. For more details on kneading theory and the $*$-operator the reader is referred to [9].

We add two more theorems to our previous list of well-known results. The second theorem is a specific case of a more general result of Davis [10] (see also [25]).

Theorem 3 [16, Corollary 1]. *Let $f:J\to J$ and $g:K\to K$ be unimodal maps with the same finite kneading sequence. Then $\varprojlim(J,f)$ and $\varprojlim(K,g)$ are homeomorphic.*

Theorem 4. *If $f:I\to I$ is full-unimodal, then $\varprojlim(I,f)$ is the Brouwer–Janiszewski–Knaster indecomposable continuum.*

For a description of the Brouwer–Janiszewski–Knaster (B–J–K) indecomposable continuum see [22, 2.9, p. 3].

4. Schwarzian derivative

The *Schwarzian derivative* of a mapping $f : I \to I$ is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

For a $\mathcal{C}^3$unimodal map f with critical point c we say f has *negative Schwarzian derivative*, denoted by $Sf < 0$, if $Sf(x) < 0$ for all $x \in I - \{c\}$. We will call any unimodal map with negative Schwarzian derivative *S-unimodal*. This class includes the *logistic family*

$$f_\lambda(x) = \lambda x(1-x) \quad \text{for } 0 < \lambda \leqslant 4$$

and

$$g_\lambda(x) = \lambda \sin(\pi x) \quad \text{for } 0 < \lambda < 1.$$

Now suppose f and g are topologically conjugate unimodal maps with critical points c and c', respectively. If h is the conjugating homeomorphism, $h(c) = c'$. Since h must be order preserving, $x < c$ if and only if $h(x) < c'$. This results in $I(x) = I(h(x))$. Therefore $k(f) = k(g)$. Although topologically conjugate unimodal maps have the same kneading sequence, it is not true that unimodal maps with the same kneading sequence are topologically conjugate. The reason for our interest in S-unimodal maps is that the kneading sequence is nearly a complete invariance of conjugacy.

The theorem below is due to Guckenheimer [14], as stated in [9], and shows that $k(f)$ determines the topological conjugacy classes except for one case in which information about stable periodic orbits is needed.

Theorem 5 [9, Theorem II.6.3, p. 126]. *Let f and g be S-unimodal maps with $k(f) = k(g) = \alpha$:*

(1) *If α is finite then f and g are topologically conjugate.*
(2) *If α is infinite and periodic of period n ($\alpha = A^\infty$, with $|A| = n$), then there are two possibilities*:
   (a) *If A is odd, then f and g are topologically conjugate if and only if their stable periodic orbits have the same period* (n *or* $2n$).
   (b) *If A is even, then f and g are topologically conjugate if and only if their stable periodic orbits* (*of period* n) *are both stable from one side or stable from both sides.*
(3) *If α is infinite and nonperiodic, then f and g are topologically conjugate.*

We will be mainly interested in the third conclusion of the theorem when we consider infinitely renormalizable unimodal maps.

5. Bennett's theorem and the core

The following theorem by Bennett [6] has proven to be very valuable in the study of inverse limits on arcs. The theorem as stated here appears in [17].

Theorem 6 (Bennett). *Suppose f is a mapping of the interval $[a, b]$ onto itself and d is a number between a and b such that*

(1) $f([d, b]) \subseteq [d, b]$;
(2) $f_{|[a,d]}$ *is monotone*; *and*
(3) *there is a positive integer k such that* $f^k([a, d]) = [a, b]$.

Then $\varprojlim([a, b], f)$ *is the union of a topological ray R and a continuum K such that* $\overline{R} - R = K$.

When f is unimodal with $f([f^2(c), f(c)]) \subseteq [f^2(c), f(c)]$, then $K = \varprojlim([f^2(c), f(c)], f)$ and is called the *core* of $\varprojlim(I, f)$. We note that if $f^3(c) = f^2(c) \neq f(c)$, then $\varprojlim(I, f)$ is the B–J–K continuum with an attached arc.

Example 1. If f is unimodal with $k(f) = RC$, then $f^2(c) = c$ and $f_{|[f^2(c), f(c)]}$ is a homeomorphism Thus, by Theorem 1 the core of $\varprojlim(I, f)$ is an arc and by Bennett's theorem, $\varprojlim(I, f)$ is a topological $\sin(\frac{1}{x})$-curve.

Now that we have reduced the study of $\varprojlim(I, f)$ to a study of the core, we can focus our attention on the nature of the core. We restrict our attention to the case where f has a fixed point $p \in (c, f(c))$ (if there is no such p then the inverse limit is an arc or a point). When the core is decomposable we make use of the following well-known result. The proof is included for completeness.

Theorem 7. *Suppose f is unimodal and the core of* $\varprojlim(I, f)$ *is decomposable. Then the core is the union of two homeomorphic subcontinua intersecting in a point or an arc.*

Proof. Let p be the fixed point for f in $(c, f(c))$ and q be the first fixed point for f^2 in $[c, f(c)]$. Then $f^3(c) \geqslant q$ and $\varprojlim([f^2(c), f(c)], f)$ is the union of $\varprojlim([f^2(c), f(q)], f^2)$ and $\varprojlim([q, f(c)], f^2)$ (see [17, Theorem 7]). Note that $f_{|[f(q), f(c)]}$ is a homeomorphism onto $[f^2(c), q]$. It follows that $f^2_{|[f(q), f(c)]}$ is topologically conjugate to $f^2_{|[f^2(c), q]}$ via $f_{|[f(q), f(c)]}$. Thus, $\varprojlim([f^2(c), f(q)], f^2)$ is homeomorphic to $\varprojlim([q, f(c)], f^2)$. Furthermore,

$$\varprojlim\bigl(\bigl[f^2(c), f(q)\bigr], f^2\bigr) \cap \varprojlim\bigl(\bigl[q, f(c)\bigr], f^2\bigr) = \varprojlim\bigl(\bigl[f(q), q\bigr], f^2\bigr)$$

produces an arc if $q \neq p$ or a point otherwise. $\square$

6. Renormalization and decomposability

Suppose f is unimodal and satisfies $f^3(c) \geqslant p$. Then the core of $\varprojlim(I, f)$ is decomposable [17, Theorem 7] and, by Theorem 7, the core decomposes into two homeomorphic subcontinua intersecting at a common endpoint of a ray. We now introduce the renormalization operator used by Feigenbaum [13] and see how it can be used to study the core.

Let $r \in f^{-1}(p)$ with $r < c$, and $s \in f^{-1}(r)$ with $s > c$. By considering f^2 we conclude that there is an interval on which f^2 is unimodal but upside-down. This observation motivates the following definition. The *renormalization of f*, denoted $\Re f$, is defined by $\Re f = h \circ f^2 \circ h^{-1}$ where $h : [r, p] \to I$ is a linear homeomorphism such that $h(r) = 1$ and $h(p) = 0$. The definition requires that $f(c) > c$ (otherwise h does not exist). We also require that $f^2([r, p]) \subseteq [r, p]$ (otherwise $\Re f$ is not unimodal). If $\Re f$ is defined then it is easy to show that $\Re f : I \to I$ is a unimodal map with critical point $h(c)$. Also if $Sf < 0$, then $S\Re f < 0$. Notice that when defined, $\Re^n f = h \circ f^{2n} \circ h^{-1}$. We note that for the *logistic family* $\Re f_\lambda$ is defined for $2 < \lambda \leqslant \lambda_1' \approx 3.6784$ (see Section 7).

Since $\Re f : I \to I$ is topologically conjugate to f^2 restricted to $[r, p]$, $\varprojlim(I, \Re f)$ is homeomorphic to $\varprojlim([r, p], f^2_{|[r,p]})$ by Theorem 2. The following is little more than a restatement of Theorem 7.

Theorem 8. *If f is unimodal with $\Re f$ defined, then the core of $\varprojlim(I, f)$ is the union of two copies of $\varprojlim(I, \Re f)$ intersecting in a point.*

Proof. Note $f^2([r, q]) = [f^2(c), f(c)]$. Thus,

$$\varprojlim\big(\big[f^2(c), f(c)\big], f\big) = \varprojlim\big([r, s], f_{|[r,s]}\big) = \varprojlim\big([r, p], f^2_{|[r,p]}\big) \cup \varprojlim\big([p, s], f^2_{|[p,s]}\big)$$

with $\varprojlim([r, p], f^2_{|[r,p]})$ homeomorphic to $\varprojlim([p, s], f^2_{|[p,s]})$. $\square$

Corollary 1. *If f is unimodal with $\Re^n f$ defined for some $n > 0$, then, for $0 \leqslant i < n$, the core of $\varprojlim(I, \Re^i f)$ is a ray limiting on the union of two copies of $\varprojlim(I, \Re^{i+1} f)$ intersecting in a point.*

Proof. Follows inductively from Theorem 8. $\square$

We use $\varprojlim(I, \Re f)$ for studying the core instead of $\varprojlim([r, p], f^2_{|[r,p]})$ so that we can take advantage of how the kneading sequences of f and $\Re f$ are related. The following result appears in [11, p. 146] (see also [12, p. 482]).

Lemma 1. *If f is unimodal with $\Re f$ defined and $k(f) = a_1 a_2 \cdots$, then $k(\Re f) = \hat{a}_2 \hat{a}_4 \cdots$. In particular, $k(f) = R * k(\Re f)$.*

Example 2. If f is unimodal with $k(f) = RLRC = R * RC$, then $k(\Re f) = RC$ by Lemma 2. It follows from Example 1 that $\varprojlim(I, \Re f)$ is a topological $\sin(\frac{1}{x})$-curve. Thus,

$\varprojlim(I, f)$ is ray limiting on two topological $\sin(\frac{1}{x})$-curves intersecting at the endpoints of their rays.

If $\Re^n f$ exists for all $n \geqslant 1$, f is said to be infinitely renormalizable. In this case, Lemma 1 shows that the kneading sequence of f is completely determined (see also Lemma 2). In recent years there has developed a more general concept of an infinitely renormalizable map which we do not consider. See [21] for more details.

7. Full families

Let **C** represent the class of $\mathcal{C}^1$ unimodal maps and let $\{f_\lambda\colon \alpha \leqslant \lambda \leqslant \beta\}$ represent a curve in **C** continuous in the $\mathcal{C}^1$ topology. More precisely, the map $\lambda \mapsto f_\lambda$ is a map from $[\alpha, \beta]$ to **C** such that

$$\lim_{\lambda\to\lambda_0}\left\{\sup_{0\leqslant x\leqslant 1}\left(\left|f_\lambda(x) - f_{\lambda_0}(x)\right| + \left|f'_\lambda(x) - f'_{\lambda_0}(x)\right|\right)\right\} = 0.$$

We say that $\{f_\lambda\colon \alpha \leqslant \lambda \leqslant \beta\}$ is a *Full family* if $k(f_\alpha) \equiv L^\infty$ and $f_\beta(c_\beta) = 1$, where c_β denotes the critical point of f_β. If $S(f_\lambda) < 0$ for all $\alpha \leqslant \lambda \leqslant \beta$, we call $\{f_\lambda\colon \alpha \leqslant \lambda \leqslant \beta\}$ an S-Full family. For convenience we sometimes write $\{f_\lambda\}$ for $\{f_\lambda\colon \alpha \leqslant \lambda \leqslant \beta\}$. The logistic family $\{f_\lambda(x) = \lambda x(1-x), 0 \leqslant \lambda \leqslant 4\}$ is an example of an S-Full family. For an example of a Full family that is not an S-Full family see [9, p. 186]. To simplify the discussion that follows we assume there are no intervals in the parameter space for which the corresponding f_λ's are infinitely renormalizable.

Theorem 9 [9, Proposition III.1.2, p. 174]. *In a Full family $\{f_\lambda\colon \alpha \leqslant \lambda \leqslant \beta\}$ every maximal sequence of the form $R\cdots$ occurs as the kneading sequence for some $\lambda \in [\alpha, \beta]$.*

Theorem 10 [24, Theorem 1.1]. *If $\{f_\lambda\colon \alpha \leqslant \lambda \leqslant \beta\}$ is a Full family, then there exists parameter values*

$$\alpha < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda'_3 < \lambda'_2 < \lambda'_1 < \beta$$

such that c_{λ_n} is periodic of period 2^n, $\Re^n f_{\lambda'_n}$ is full-unimodal, and $\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\lambda'_n$.

The limiting parameter value $\lambda_\infty = \lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\lambda'_n$ is called the *Feigenbaum value*.

We note that f_{λ_∞} is infinitely renormalizable.

Lemma 2. *Let $\{f_\lambda\}$ be a Full family with sequences $\{\lambda_n\}$ and $\{\lambda'_n\}$ as above. Then*

(1) $k(f_{\lambda_1}) = RC$ *and* $k(f_{\lambda_n}) = (R*)^{n-1}RC = R * k(f_{\lambda_{n-1}})$ *for all* $n > 1$,
(2) $k(f_{\lambda'_1}) = RLR^\infty$ *and* $k(f_{\lambda'_n}) = (R*)^{n-1}RLR^\infty = R * k(f_{\lambda'_{n-1}})$ *for all* $n > 1$,
(3) $k(f_{\lambda_\infty}) = \lim_{n\to\infty} k(f_{\lambda_n}) = (R*)^\infty RC$ *is nonperiodic.*

Proof. (1) Follows from [9, Lemma II.2.12, p. 79]. (2) Let $n > 1$ and suppose $\Re^n f_{\lambda'_n}$ is full-unimodal. Then $k(\Re^n f_{\lambda'_n}) = RL^\infty$. Repeatedly applying Lemma 1 results in

$$\begin{aligned} k(f_{\lambda'_n}) &= R * k(\Re f_{\lambda'_n}) = (R*)^2 k\big(\Re^2 f_{\lambda'_n}\big) = \cdots = (R*)^n k\big(\Re^n f_{\lambda'_n}\big) \\ &= (R*)^n RL^\infty = (R*)^{n-1} RLR^\infty = R * k(f_{\lambda'_{n-1}}). \end{aligned}$$

(3) Suppose $k(f_{\lambda_\infty}) = (b_0 b_1 \cdots b_k)^\infty$. Then $R * (b_0 b_1 \cdots b_k)^\infty = (R\hat{b}_0 R\hat{b}_1 \cdots R\hat{b}_k)^\infty = (b_0 b_1 \cdots b_k)^\infty$ implies that $b_k = \hat{b}_k$, a contradiction. □

Before considering the parameter values discussed above, we make the following general observations.

Theorem 11. *Let f_λ be a member of a Full family $\{f_\lambda\}$ with $k(f_\lambda) = BC$. Then there exists an $\varepsilon > 0$ such that for all $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$, $\varprojlim(I, f_\mu)$ and $\varprojlim(I, f_\lambda)$ are homeomorphic.*

Proof. Suppose f_λ be a member of a Full family $\{f_\lambda\}$ with $k(f_\lambda) = BC$ and $|BC| = n$. According to [9, Lemma III.1.3, p. 174] there exists an $\varepsilon > 0$ such that $\forall \mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$, $k(f_\mu) \in \{(BL)^\infty, BC, (BR)^\infty\}$. Furthermore, as in the proof of Theorem 16.4.2 in [20], ε can be chosen so that c_μ is in the immediate basin of an attracting periodic point p of period n for f_μ. If $k(f_\mu) = BC$, then $\varprojlim(I, f_\mu)$ and $\varprojlim(I, f_\lambda)$ are homeomorphic (Theorem 3). If $k(f_\mu) \in \{(BL)^\infty, (BR)^\infty\}$, then let A_1 be a closed interval containing c_μ and contained in the immediate basin of p. Define $A_2 = f_\mu(A_1)$, $A_3 = f_\mu(A_2), \ldots, A_n = f_\mu(A_{n-1})$. It follows that f_μ is a nearly Markov map on $J = [f_\mu^2(c_\mu), f_\mu(c_\mu)]$ with respect to $A_1, A_2, \ldots, A_n$ (see [2]). By Lemma 2.3 in [2], f_μ can be modified to construct a unimodal map f with $k(f) = BC$ satisfying the conditions of Theorem 2.1 in [2]. We conclude that $\varprojlim(I, f)$ and $\varprojlim(I, f_\mu)$ are homeomorphic. It follows that $\varprojlim(I, f_\mu)$ and $\varprojlim(I, f_\lambda)$ are homeomorphic. □

If $k(f_\lambda) = BC$, then the core of $\varprojlim(I, f_\lambda)$ has n endpoints. It follows that for parameters μ in the prior result, the core of $\varprojlim(I, f_\mu)$ has $n = |\omega_{f_\mu}(c_\mu)|$ endpoints, where $\omega_{f_\mu}(c_\mu) = \{x \in I\colon \exists$ a sequence $n_k \to \infty$ with $f_\mu^{n_k}(c_\mu) \to x\}$. This observation combined with an argument similar to the previous proof can be used to obtain the following result. As mentioned above the case that $k(f) = k(g) = BC$ was proved by Holte [16].

Theorem 12. *Let f and g be $\mathcal{C}^1$ unimodal maps with critical points c_f and c_g, respectively. Suppose $k(f) = k(g) \in \{(BL)^\infty, BC, (BR)^\infty\}$. Then $\varprojlim(I, f)$ and $\varprojlim(I, g)$ are homeomorphic if and only if $|\omega_f(c_f)| = |\omega_g(c_g)|$.*

8. Below the Feigenbaum value

In this section we consider parameter values with corresponding kneading sequences below the Feigenbaum value, λ_∞.

Theorem 13. *Let $\{f_\lambda\}$ be a Full family with sequence $\{\lambda_n\}$ as in Theorem* 10. *Then $\varprojlim(I, f_{\lambda_1})$ is a topological $\sin(\frac{1}{x})$-curve and the core of $\varprojlim(I, f_{\lambda_{n+1}})$ is the union of two copies of $\varprojlim(I, f_{\lambda_n})$ intersecting in a point. Furthermore, for each n there exists an ε_n such that for all $\lambda \in (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$, $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$.*

Proof. Let $n > 1$. Since $\Re f_{\lambda_{n+1}}$ exists, it follows by Theorem 7 that the core of $\varprojlim(I, f_{\lambda_{n+1}})$ is the union of two copies of $\varprojlim(I, \Re f_{\lambda_{n+1}})$ intersecting in a point. By (1) of Lemma 2, $k(\Re f_{\lambda_{n+1}}) = k(f_{\lambda_n}) = (R*)^{n-1}RC$. By Theorem 3, $\varprojlim(I, \Re f_{\lambda_{n+1}})$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$. Since $k(f_{\lambda_1}) = RC$, $\varprojlim(I, f_{\lambda_1})$ is a topological $\sin(\frac{1}{x})$-curve (see Example 1). The remaining conclusion of the theorem now follows from Theorem 11. □

Since $\varprojlim(I, f_{\lambda_1})$ is a topological $\sin(\frac{1}{x})$-curve, it follows that $\varprojlim(I, f_{\lambda_2})$ is a ray limiting on the union of two topological $\sin(\frac{1}{x})$-curves intersecting the endpoint of their rays. Also, $\lim(I, f_{\lambda_3})$ is a ray limiting on the union of two rays intersecting an endpoint each limiting on two topological $\sin(\frac{1}{x})$-curves intersecting at the endpoint of their rays. In general, $\varprojlim(I, f_{\lambda_n})$ is a ray limiting on the union of two rays intersecting in a common endpoint each limiting on the union of two rays intersecting in a common endpoint... limiting on the union of two rays intersecting an endpoint each limiting on two topological $\sin(\frac{1}{x})$-curves intersecting the endpoint of their rays. Thus, $\varprojlim(I, f_{\lambda_n})$ contains 2^{n-1} topological $\sin(\frac{1}{x})$-curves. In fact, as is shown below, for any λ with $k(f_\lambda) < k(f_{\lambda_\infty})$, $\varprojlim(I, f_\lambda)$ is a point, an arc, or there exists a λ_n such that $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$.

Theorem 14. *Let $\{f_\lambda\}$ be a Full family with sequence $\{\lambda_n\}$ as in Theorem* 10. *For all λ with $k(f_\lambda) < k(f_{\lambda_\infty})$, $\varprojlim(I, f_\lambda)$ is a point, an arc, or there exists an n such that $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$.*

Proof. Let $k(f_\lambda) < k(f_{\lambda_\infty})$. If $k(f_\lambda) \leqslant R^\infty$ and $|w_{f_\lambda}(c_\lambda)| = 1$, then $\varprojlim(I, f_\lambda)$ is a point or an arc. If $R^\infty \leqslant k(f_\lambda) < k(f_{\lambda_\infty})$ and $|w_{f_\lambda}(c_\lambda)| \geqslant 2$, then there exists an n such that $k(f_\lambda) \in \{(R*)^n R^\infty, (R*)^n RC, (R*)^{n+1} R^\infty\}$ [9, Lemma II.2.12, p. 79]. If $k(f_\lambda) = (R*)^n RC$ then, since $k(f_{\lambda_{n+1}}) = (R*)^n RC$, $\varprojlim(I, f_{\lambda_{n+1}})$ is homeomorphic to $\varprojlim(I, f_\lambda)$ by Theorem 3. If $k(f_\lambda) = (R*)^n R^\infty$ (the argument for the other case is handled similarly). Then, either $|w_{f_\lambda}(c_\lambda)| = 2^n$ or $|w_{f_\lambda}(c_\lambda)| = 2^{n+1}$ [9, Lemma II.3.2, p. 83]. If $|w_{f_\lambda}(c_\lambda)| = 2^n$, then, as in the proof of Theorem 11, there exists a parameter value μ such that $k(f_\mu) = (R*)^n R^\infty$, $|w_{f_\mu}(c_\mu)| = 2^n$, and $\varprojlim(I, f_\mu)$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$. Since $k(f_\lambda) = k(f_\mu)$ is periodic of period 2^n and $|w_{f_\lambda}(c_\lambda)| = |w_{f_\mu}(c_\mu)| = 2^n$, $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_\mu)$ by Theorem 12. Thus, $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\lambda_n})$. By a similar argument, if $|w_{f_\lambda}(c_\lambda)| = 2^{n+1}$, then $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\lambda_{n+1}})$. □

As a result, we see that the topology of $\varprojlim(I, f_\lambda)$ with $k(f_\lambda) < k(f_{\lambda_\infty})$ is completely determined with a fascinating pattern in the appearance of topological $\sin(\frac{1}{x})$-curves as we get closer to the Feigenbaum value, λ_∞.

Example 3. Consider the logistic family $f_\lambda(x) = \lambda x(1-x)$ for $0 \leqslant \lambda \leqslant 4$. Note that $c_\lambda = \frac{1}{2}$ for all λ. As we have already noted, there is an increasing sequence of parameter values

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

at which the logistic map f_{λ_n} has periodic critical point of period 2^n. These values are in fact unique since the map $\lambda \mapsto k(f_\lambda)$ is monotone [21, Theorem 10.1, p. 194 and Corollary 1, p. 197]. There also exists an unique sequence of period-doubling bifurcation values $\{\mu_n\}_{n\geqslant 0}$ such that $\lambda_n < \mu_n < \lambda_{n+1}$ (see [9]). This sequence is such that, for all $\lambda \in (\mu_n, \mu_{n+1}]$, $|\omega_{f_\lambda}(\frac{1}{2})| = 2^{n+1}$. Furthermore, with $k(f_{\lambda_{n+1}}) = (R*)^n RC = BC$, $k(f_\lambda) \in \{(BL)^\infty, BC, (BR)^\infty\}$ for all $\lambda \in (\mu_n, \mu_{n+1}]$. By Theorem 12, $\varprojlim(I, f_\lambda)$ and $\varprojlim(I, f_\mu)$ are homeomorphic for all $\lambda, \mu \in (\mu_n, \mu_{n+1}]$. By Theorem 13, $\varprojlim(I, f_\lambda)$ is a topological $\sin(\frac{1}{x})$-curve for all $\lambda \in (\mu_0, \mu_1]$ and, for all $\lambda \in (\mu_n, \mu_{n+1}]$, the core of $\varprojlim(I, f_\lambda)$ is the union of two copies of $\varprojlim(I, f_\mu)$ intersecting at a common endpoint of a ray for any $\mu \in (\mu_{n-1}, \mu_n]$.

9. Above the Feigenbaum value

In this section we are unable to get a complete classification of the possible inverse limits as we did for parameter values below the Feigenbaum value. However, we still are able to reveal some interesting patterns occurring.

The following result is observed for the specific case of $BC = RLC$ in [12]. We assume the general statement is known although we could not find it in the literature.

Theorem 15. *Let $\{f_\lambda\}$ be a Full family and $BC > k(f_{\lambda_\infty})$ a maximal sequence and $\{\lambda'_n\}$ as in Theorem* 10. *Then there exists a decreasing sequence $\{\mu_n\}$ converging to λ_∞ such that $k(f_{\mu_n}) = (R*)^n BC$. Moreover, there exists an $m \geqslant 1$ such that $\lambda'_{m+n+1} < \mu_n < \lambda'_{m+n}$ for all $n \geqslant 1$.*

Proof. Since $\{f_\lambda\}$ is a Full family and $BC > k(f_{\lambda_\infty})$ is maximal, by Theorem 9 there exists parameter value $\mu_0 > \lambda_\infty$ such that $k(f_{\mu_0}) = BC$. Note that μ_0 is not necessarily unique. Also, $(R*)^n BC$ is maximal [9, Corollary II.2.4, p. 75] and $k(f_{\lambda_\infty}) = (R*)^\infty BC$. Since the $*$-operator is order preserving on maximal sequences [9, Theorem II.2.5, p. 75] and $k(f_{\lambda_\infty}) < BC < RL^\infty$, we have $(R*)^\infty BC < R * BC < R * RL^\infty = k(f_{\lambda'_1})$. Thus, there exists an $m \geqslant 1$ such that

$$k(f_{\lambda'_{m+1}}) = (R*)^{m+1} RL^\infty < R * BC < (R*)^m RL^\infty = k(f_{\lambda'_m}).$$

It follows again from Theorem 9 that there exists $\mu_1 \in (\lambda'_{m+1}, \lambda'_m)$ such that $k(f_{\mu_1}) = R * BC$. Again note that μ_1 is not necessarily unique. Again it follows that

$$(R*)^{m+2}(RL)^\infty < (R*)^2 BC < (R*)^{m+1}(RL)^\infty.$$

and there exists $\mu_2 \in (\lambda'_{m+2}, \lambda'_{m+1})$ such that $k(f_{\mu_2}) = (R*)^2 BC$. Continuing in this manner we obtain a sequence $\{\mu_n\}$ with the desired properties. $\square$

Theorem 16. *Let $\{f_\lambda\}$ be a Full family, $BC > k(f_{\lambda_\infty})$ a maximal sequence, and $\{\mu_n\}$ the sequence from Theorem 15. Then the core of $\varprojlim(I, f_{\mu_n})$ is the union of two copies of $\varprojlim(I, f_{\mu_{n-1}})$ intersecting in a common endpoint of a ray. Moreover, for each μ_n there exists ε_n such that for all $\lambda \in (\mu_n - \varepsilon_n, \mu_n + \varepsilon_n)$, $\varprojlim(I, f_\lambda)$ is homeomorphic to $\varprojlim(I, f_{\mu_n})$.*

Proof. By Theorem 9, the core of $\varprojlim(I, f_{\mu_n})$ is the union of two copies of $\varprojlim(I, \Re f_{\mu_n})$ intersecting in a common endpoint of a ray. But $k(f_{\mu_{n-1}}) = k(\Re f_{\mu_n}) = (R*)^{n-1} BC$ is of finite length. Therefore, by Theorem 3, $\varprojlim(I, f_{\mu_{n-1}})$ is homeomorphic to $\varprojlim(I, \Re f_{\mu_n})$. The remaining part of the conclusion of the theorem now follows from Theorem 11. $\square$

Suppose BC is maximal, $BC > k(f_{\lambda_\infty})$, and $|BC| = k$. Then $\varprojlim(I, f_{\mu_0})$ is an indecomposable continuum with $k + 1$ endpoints, k of which are contained in the core. For example, if $BC = RLC$, then the core is the classical three-endpoint indecomposable continuum (see [3, Theorem 8]). At parameter value μ_1, $\varprojlim(I, f_{\mu_1})$ is a ray limiting on the union of two copies of $\varprojlim(I, f_{\mu_0})$ intersecting in a common endpoint of a ray. At parameter value μ_2, $\varprojlim(I, f_{\mu_2})$ is a ray limiting on the union of two rays intersecting a common endpoint each limiting on the union of two copies of $\varprojlim(I, f_{\mu_1})$ intersecting in a common endpoint of a ray. In general, $\varprojlim(I, f_{\mu_n})$ is a ray limiting on the union of two rays intersecting in a common endpoint each limiting on the union of two rays intersecting in a common endpoint... limiting on the union of two rays intersecting an endpoint each limiting on two copies of $\varprojlim(I, f_{\mu_0})$ intersecting the endpoint of their rays. Thus, $\varprojlim(I, f_{\mu_n})$ contains 2^n copies of $\varprojlim(I, f_{\mu_0})$. This situation is completely analogous to the behavior below λ_∞, with each topological $\sin(\frac{1}{x})$-curves replaced by $\varprojlim(I, f_{\mu_0})$. In particular, if each topological $\sin(\frac{1}{x})$-curve in $\varprojlim(I, f_{\lambda_n})$ is replaced by $\varprojlim(I, f_{\mu_0})$, the result is $\varprojlim(I, f_{\mu_{n-1}})$.

Example 4. Let $\{f_\lambda\}$ be a Full family with $BC = RLC$. Theorem 15 guarantees a decreasing sequence $\{\mu_n\} \to \lambda_\infty$ with $\mu_0 > \lambda'_1$ and $k(f_{\mu_0}) = RLC$. Then $\varprojlim(I, f_\mu)$ is the well-known three endpoint indecomposable continuum. Therefore, $\varprojlim(I, f_{\mu_n})$ is a ray limiting on the union of two rays intersecting an endpoint each limiting on two rays intersecting an endpoint... limiting on two rays intersecting an endpoint each limiting on the three endpoint continuum.

Theorem 17. *Let $\{f_\lambda\}$ be a Full family with the sequence $\{\lambda'_n\}$ from Theorem 10. Then the core of $\varprojlim(I, f_{\lambda'_1})$ is the union of two B–J–K continua intersecting in a common endpoint of a ray and the core of $\varprojlim(I, f_{\lambda'_n})$ is the union of two copies of $\varprojlim(I, f_{\lambda'_{n-1}})$ intersecting in a common endpoint of a ray.*

Proof. By Theorem 10 $\Re^n f_{\lambda'_n}$ is full-unimodal. By Theorem 6 $\varprojlim(I, \Re^n f_{\lambda'_n})$ is the B–J–K continuum. Now apply Corollary 1. $\square$

We note that since λ'_n is on the boundary of where $\Re^n f_{\lambda'_n}$ exists, we cannot hope to get results equivalent to the second part of Theorem 13. Also, unlike the case where every inverse limit with bonding map having kneading sequence less than $k(f_{\lambda_\infty})$ contains copies of $\sin(\frac{1}{x})$-curves, we are unable to classify all inverse limits occurring with bonding map from above the Feigenbaum value. Perhaps the broadest statement we can make is the following.

Theorem 18. *Let $\{f_\lambda\}$ be a Full family with Feigenbaum value λ_∞. Then for all λ with $k(f_\lambda) > k(f_{\lambda_\infty})$, $\varprojlim(I, f_\lambda)$ contains an indecomposable continuum.*

Proof. If $k(f_\lambda) > k(f_{\lambda_\infty})$ then there exists a smallest n such that $\Re^n f_\lambda$ does not exist. Thus, there exists an interval $[a, b]$ containing the critical point in its interior such that $[a, b] \subsetneq f_\lambda^{2^n}([a, b])$. In other words, $f_\lambda^{2^n}$ contains a horseshoe. The result follows from [8, Lemma 3] and [1, Corollary 11]. $\square$

10. The Feigenbaum value

In this section we identify the inverse limit occurring at the Feigenbaum value under certain smoothness conditions. Since any two infinitely renormalizable unimodal maps have the same nonperiodic kneading sequence, we state the theorems in this section for infinitely renormalizable unimodal maps in general instead of those corresponding to a Full family. As mentioned earlier, the results of this section were proved previously (using different arguments) by Ingram and Roe in [18].

Suppose f is an infinitely renormalizable unimodal map. Noting that $k(\Re^n f) = k(f)$, we would like to conclude $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, \Re^n f)$. However, $k(f)$ is infinite and nonperiodic so we cannot use Theorem 2. However, by limiting ourselves to S-unimodal maps we can apply Theorem 5 to conclude that $\Re^n f$ is topologically conjugate to f. Theorem 2 then shows that $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, \Re^n f)$. In addition, if g is any other infinitely renormalizable S-unimodal map, then $k(f) = k(g)$ so that f and g are topologically conjugate. Again, Theorem 2 shows that $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, g)$. We have proved the following.

Theorem 19. *If f is an infinitely renormalizable S-unimodal map, then $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, \Re^n f)$ for all $n \geqslant 1$.*

Theorem 20. *If f and g are infinitely renormalizable S-unimodal maps, then $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, g)$.*

If we only consider infinitely renormalizable S-unimodal maps f and g, then we are forcing the dynamics of the induced homeomorphisms $\hat{f} : \varprojlim(I, f) \to \varprojlim(I, f)$ and $\hat{g} : \varprojlim(I, g) \to \varprojlim(I, g)$ to be identical. Of course two bonding maps need not be topologically conjugate in order to produce homeomorphic inverse limits. We now extend Theorem 20 to a larger class of infinitely renormalizable maps where the induced homeomorphisms are not necessarily topologically conjugate.

Following [21], we call the critical point c for a $\mathcal{C}^2$ map f *non-flat* if there exists a $\mathcal{C}^2$ local diffeomorphism ϕ with $\phi(c)=0$ such that $f(x)=\pm|\phi(x)|^\alpha+f(c)$ for some $\alpha\geqslant 2$. Note any $\mathcal{C}^2$ map f which is $\mathcal{C}^{k+1}$ in a neighborhood of c with $f^{(k)}(c)\neq 0$ for some $k\geqslant 2$ implies c is non-flat. Also, note that since we are only considering unimodal maps, f is of the form $f(x)=-|\phi(x)|^\alpha+f(c)$.

Theorem 21. *Suppose f and g are infinitely renormalizable $\mathcal{C}^2$ unimodal maps with non-flat critical points. Then $\varprojlim(I,f)$ is homeomorphic to $\varprojlim(I,g)$.*

Proof. For $a,b\in I$, $[a,b]$ will denote the smallest closed interval containing a and b, while (a,b) will denote the interior of $[a,b]$. The sets

$$\omega_f(c_f)=\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{n}\bigl[f^i(c_f),f^{i+2^n}(c_f)\bigr]\quad\text{and}$$
$$\omega_g(c_g)=\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{n}\bigl[g^i(c_g),g^{i+2^n}(c_g)\bigr]$$

are Cantor sets ([21, Theorem 6.2, p. 156 and Proposition 4.5, p. 242], [15, p. 346]). Because $k(f)=k(g)$ is infinite and nonperiodic, $I(f^i(c_f))=I(g^i(c_g))\neq I(f^j(c_f))=I(g^j(c_g))$ for all $i\neq j\geqslant 0$. It follows that the sets $\{c_f,f(c_f),f^2(c_f),\ldots\}$ and $\{c_g,g(c_g),g^2(c_g),\ldots\}$ have the same ordering in I. We construct an order preserving homeomorphism $h:I\to I$ with $h(f^i(c_f))=g^i(c_g)$ for all $i\geqslant 0$ as follows.

Let

$$A_n=\bigcup_{i=1}^{n}\bigl[f^i(c_f),f^{i+2^n}(c_f)\bigr]\quad\text{and}$$
$$B_n=\bigcup_{i=1}^{n}\bigl[g^i(c_g),g^{i+2^n}(c_g)\bigr]$$

and define $h_1:I\to I$ as the (unique) order preserving piecewise-linear homeomorphism such that $h(A_1)=B_1$. Since $\cdots\subseteq A_3\subseteq A_2\subseteq A_1$ and $\cdots\subseteq B_3\subseteq B_2\subseteq B_1$ are such that $A_{n-1}-A_n$ and $B_{n-1}-B_n$ consist of 2^n open intervals, one interval from each of the interiors of $[f^i(c_f),f^{i+2^n}(c_f)]$ and $[g^i(c_g),g^{i+2^n}(c_g)]$, $1\leqslant i\leqslant 2^n$, we can inductively define $h_n:I\to I$ as $h_n(x)=h_{n-1}(x)$ for all $x\notin A_{n-1}-A_n$ and h_n is order preserving piecewise-linear on $A_{n-1}-A_n$ such that $h_n(A_{n-1}-A_n)=B_{n-1}-B_n$. Each h_n is continuous and $h_n(f^i(c_f))=g^i(c_g)$ for $0\leqslant i\leqslant 2^n$. Using the fact that the diameters of $[f^i(c_f),f^{i+2^n}(c_f)]$ and $[g^i(c_g),g^{i+2^n}(c_g)]\to 0$ as $n\to\infty$, we conclude h_n converges uniformly to a continuous function h, which is onto and $h(f^i(c_f))=g^i(c_g)$ for all $0\leqslant i$. It remains to show that h is one-to-one. If h is not one-to-one it must be monotone on some interval $[x,y]$. Since $\omega_f(c_f)$ is perfect and $h(f^i(c_f))\neq h(f^j(c_f))$ for all $i\neq j$, $f^i(c_f)\notin(x,y)$ for all $0\leqslant i$. Therefore, there exist some $i\neq j$ and $k\geqslant 0$ such that $[x,y]\subseteq[f^i(c_f),f^j(c_f)]$ and $[f^i(c_f),f^j(c_f)]\cap A_k=\{f^i(c_f),f^j(c_f)\}$. But h was constructed so that h restricted to $[f^i(c_f), f^j(c_f)]$ is equal to h_k restricted $[f^i(c_f), f^j(c_f)]$. Since h_k is a homeomorphism $h(x) \neq h(y)$, a contradiction. It follows that h is an order preserving homeomorphism. It follows from [5, Lemma 1.3] that $\varprojlim(I, f)$ is homeomorphic to $\varprojlim(I, g)$. □

Since every member f_λ from the logistic family has $Sf_\lambda < 0$ and non-flat critical point, we see that the inverse limits considered in Theorems 20 and 21 are homeomorphic.

We now turn our attention to the topological properties of the inverse limits of the previous theorem. Recall that if f is infinitely renormalizable with $Sf < 0$, then $\varprojlim(I, f)$ and $\varprojlim(I, \Re^n f)$ are homeomorphic for all $n \geqslant 1$. Thus the core of $\varprojlim(I, \Re^n f)$ consists of two copies of $\varprojlim(I, f)$ intersecting at a common endpoint of a ray. This observation suggests some restrictions on the types of topologically different subcontinua of $\varprojlim(I, f)$. This is precisely what was observed by Barge and Ingram for the logistic family [3, Theorem 7]. In fact, their proof for the logistic family can be used without any modification for any infinitely renormalizable map f with $Sf < 0$.

Theorem 22. *Suppose f is an infinitely renormalizable unimodal map with either $Sf < 0$ or non-flat critical point. Then $\varprojlim(I, f)$ is hereditarily decomposable and contains only three topologically different subcontinua*: *arcs, copies of $\varprojlim(I, f)$, or the union of two copies of $\varprojlim(I, f)$ intersecting in a point.*

Proof. Follows from [3, Theorem 7] and the fact that every member f_λ from the logistic family has $Sf_\lambda < 0$ and non-flat critical point. □

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